

NASA 532-63  
188200  
108

# Manipulator Control by Exact Linearization

K. Kreutz

Jet Propulsion Laboratory  
California Institute of Technology  
Pasadena, CA 91109

77 572-50

## 1. Abstract.

Comments on the application to rigid link manipulators of Geometric Control Theory, Resolved Acceleration Control, Operational Space Control, and Nonlinear Decoupling Theory are given, and the essential unity of these techniques for externally linearizing and decoupling end effector dynamics is discussed. Exploiting the fact that the mass matrix of a rigid link manipulator is positive definite - a consequence of rigid link manipulators belonging to the class of natural physical systems - it is shown that a necessary and sufficient condition for a locally externally linearizing and output decoupling feedback law to exist is that the end effector Jacobian matrix be nonsingular. Furthermore, this linearizing feedback is easy to produce.

## 2. Introduction.

Because of the difficulty in controlling rigid link manipulators, along with a primary concern in controlling end effector (EF) motions, it is natural to ask if a nonlinear feedback law exists which will make an EF behave as if it has linear and decoupled dynamics. It has been known at least since the early 1970s [1]-[5] that exact linearization of manipulators in joint space is readily accomplished by the so-called Inverse or Computed Torque Technique. Efforts to accomplish decoupled linearization of EF motions directly in task space began soon thereafter as is evident in the work of [6]-[14].

The work of [6], although concerned only with controlling the tip location of a three-link manipulator in the plane, is surprisingly prescient in its approach in that it proceeds by the three explicit steps of 1) decoupled linearization of tip behavior; 2) stabilization of the resulting tip dynamics; followed by 3) trajectory control of the now linearly behaving tip. Such clarity of approach will only be retrieved in the latter work of [19]-[22]. The work [6] also presages future work in its dealing with the problems of manipulator redundancy and actuator saturation.

With hindsight, the work [6] can also be viewed as a direct precursor to the development of the Resolved Acceleration Control (RAC) approach to the end effector tracking problem [7][8]. RAC essentially extends the work of [6] to the case of a full six dof manipulator yielding linearized EF positional error dynamics and almost linearized EF attitude error dynamics (the extent to which attitude error dynamics are "almost" linearized will be discussed below). The work of [7][8], however, did not make clear the three steps of [6] and consequently appears to have not been appreciated as a technique for performing decoupled exact linearization of EF motions, but rather as a technique for end effector tracking which has (almost) linear tracking error dynamics. The fact that the attitude error dynamics are not completely linearized also apparently obscured the appreciation of RAC as an exactly linearizing control technique.

The work of [9]-[11] applies Nonlinear Decoupling Theory (NDT) to provide decoupled linearization of a manipulator EF with simultaneous pole placement of the linearized EF dynamics. The abstruse formulation of this approach has apparently discouraged serious comparison with other approaches, the notable exception being [23] where correspondences to RAC and the Computed Torque technique have been noted. The simultaneous pole placement and linearization of EF dynamics represents a blurring of the distinct steps 1 and 2 described above for the approach [6].

In [12]-[14], manipulator dynamics are expressed in the task space, or Operational Space of the EF. The resulting nonlinear effective end effector dynamics are then linearized by the Computed Torque method. Thus, the Operational Space Control (OCS) of [12]-[14] can also be viewed as a Generalized Computed Torque technique. In [12] correspondences to RAC and the Computed Torque technique have been noted.

Recently, Geometric Control Theory (GCT) based techniques for exactly externally linearizing and decoupling general affine-in-the-input nonlinear systems have been developed [15]-[19]. These techniques provide constructive sufficient conditions for local decoupled external linearization which, if satisfied, produces the linearizing feedback law. GCT has been applied to exactly linearizing end effector motions in [19]-[22]. The work of [19]-[22] also provides a clear and mature control perspective which keeps the following steps distinct: 1) Exactly linearize and decouple end effector dynamics to a canonical decoupled double integrator form, i.e. to Brunovsky Canonical Form (BCF); 2) Effect a stabilizing loop (pole placement step); 3) Perform feedforward precompensation to obtain nominal model following performance; 4) Institute an LQR error correcting feedback loop. Unfortunately, to understand the theoretical underpinnings of GCT requires

an exposure to differential geometry and Lie algebra/Lie group theory which most practicing engineers are unlikely to have.

It can be shown that all of the above seemingly quite different approaches lead to the same linearizing control law for exact external linearization and decoupling of EF motions [24]. (This equivalence is specific to the nonlinear systems considered here, viz. systems dynamically similar to rigid link manipulators. NDT and GCT apply to a much larger class than this, and so the equivalence to RAC and OSC holds for systems restricted to this class but not in general). Recognizing this equivalence enables us to give a simple necessary and sufficient condition for local decoupled external linearization and to give a simple form for the linearizing control which is applicable to a broad class of so-called natural physical dynamical systems [25] [26] of which a serial link manipulator is but a special case. For brevity we do not discuss actuated redundant arms - for discussion of these cases, see [24].

### 3. Dynamics of Finite Dimensional Natural Systems.

Many physical systems have finite dimensional nonlinear dynamics of the form [25][26]:

$$M(q)\ddot{q} + V(q, \dot{q}) = \tau; \quad q \in \mathbb{R}^n; \quad \dot{q}, \ddot{q} \in \mathbb{R}^n; \quad (1)$$

$$M(q) \in \mathbb{R}^{n \times n}; \quad M(q) = M^T(q) > 0, \quad \forall q$$

where  $q$  evolves on a manifold of dimension  $n$ . For example  $q \in \mathbb{R}^n$  for a Cartesian manipulator, while  $q \in \mathbb{T}^n$  for a revolute manipulator.

Typically (1) arises as a solution to the Lagrange equations:

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}} L - \frac{\partial L}{\partial q} = Q^T$$

where  $L = T - U$ ,  $T = 1/2 \dot{q}^T M(q) \dot{q}$  is positive definite and autonomous,  $U$  is a conservative potential function,  $Q = \tau + F$  are generalized forces, and  $F$  are dissipative or constraint forces. This is exactly how manipulator dynamics are obtained and hence manipulator dynamics are precisely of the form (1). Systems which arise in this way are known as natural systems [25][26]. It is known that for natural systems not only is  $M(q)$  positive definite, but  $V(q, \dot{q})$  of (1) has terms which depend on  $M(q)$  in a very special way [27]-[29]. In fact, natural systems are nongeneric in the class of all affine-in-the-input nonlinear systems [38][39]. Although we shall only exploit the fact that  $M(q)$  is positive definite for any  $q$ , it is worth noting that the nongeneric structure of (1) has recently enabled important statements to be made on the existence of time optimal control laws [38]-[40], on the existence of globally stable control laws [27]-[33], on the existence of robust exponentially stable control laws [34], and on the existence of stable adaptive control laws [35]-[37] for the natural system (1).

Recognizing the special properties of the system (1), it is not surprising that results yielding external linearizing behavior can be obtained much more easily than by application of NDT or GCT - theories which apply to the whole general class of smooth affine-in-the-input nonlinear systems.

### 4. End Effector Kinematics and Control After Linearization.

The system (1) is assumed to have a read-out map of either the form

$$y = h(q) \in \mathbb{R}^m, \quad \dot{y} = J(q)\dot{q} \in \mathbb{R}^m, \quad J(q) = \frac{\partial h}{\partial q} \in \mathbb{R}^{m \times n} \quad (2)$$

or of the more general form

$$y = h(q) \in \mathbb{M}^m, \quad \dot{y} = J_0(q)\dot{q} \in \mathbb{R}^m, \quad J_0(q) \in \mathbb{R}^{m \times n} \quad (3)$$

where  $J_0 dt - \dot{y} dt$  is a general, perhaps nonintegrable, Pfaffian form [25][26],  $h(\cdot)$  is  $C^2$  [44][47] and defined on  $N^n$ ,  $\mathbb{M}^m$  is some  $m$  dimensional output manifold,  $J$  or  $J_0$  is  $C^1$ , and in general  $m$  and  $n$  have different values. Often  $h(\cdot)$  is smooth (i.e.  $C^\infty$ ) or even a diffeomorphism when the domain is suitably restricted. In subsequent discussion  $\dot{y} = \bar{J}\dot{q}$  will mean that  $\bar{J}$  can be either  $J$  or  $J_0$ . Let the state of system (1) be  $(q, \dot{q})$ . Then for  $y = h(q)$ ,  $\dot{y} = J(q)\dot{q}$  will be called the "velocity associated with the output  $y$ ." Note that (2) is a special case of (3) where  $\dot{y}$ , the velocity associated with  $y$ , is just  $\dot{y} = \dot{y}$  and  $\mathbb{M}^m = \mathbb{R}^m$  giving  $\bar{J} = J = \partial h / \partial q$ . Also note that for the case (3), since  $h$  is  $C^2$ , it is still meaningful to talk about  $\dot{y} = J\dot{q}$  and  $J = \partial h / \partial q$ ,  $J(q): T_q N^n \rightarrow \mathbb{R}^m = T_h(q) \mathbb{M}^m$ , but now the case where  $\dot{y} \neq \dot{y}$  is admitted as a possibility.

For rigid link manipulators moving in Euclidean 3-space, typically  $\dot{V} = \begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} \in \mathbb{R}^6$ , where  $x \in \mathbb{R}^3$  gives the EF location,  $\dot{x}$  the EF linear velocity, and  $\omega \in \mathbb{R}^3$  the EF angular rate of change. It is well known that  $\omega$  is not the time derivative of any minimal (i.e. 3 dimensional) representation of attitude, so that  $\dot{V} = \begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} = J_0(q)\dot{q}$  as in (3). In this case, we call  $J_0(q)$  the "Standard Jacobian." It is also common to represent EF attitude by a proper orthogonal matrix  $A \in \mathbb{R}^{3 \times 3}$ ,

$$A \in \text{SO}(3) = \left\{ A \mid A^T A = A A^T = I, \det A = +1 \right\},$$

where the columns of  $A$  determine EF fixed body axes in the usual way. It is well known that  $A = \tilde{\omega} A$  where  $\tilde{\omega} v = \omega \times v$  for all  $v \in \mathbb{R}^3$ . Thus EF location and kinematics are often given by

$$y = (x, A) = h(q) \in \mathbb{R}^3 \times \text{SO}(3), \quad \dot{V} = \begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} = J_0(q)\dot{q} \in \mathbb{R}^6, \quad \dot{A} = \tilde{\omega} A, \quad A \in \text{SO}(3), \quad \tilde{\omega}^T = -\tilde{\omega}, \quad J_0(q) \in \mathbb{R}^{6 \times n}, \quad (4)$$

which should be compared to (3). Alternatively, we can take (cf. (2))

$$y = \begin{pmatrix} x \\ \beta \end{pmatrix} = h(q) \in \mathbb{R}^6, \quad \dot{V} = \dot{y} = \begin{pmatrix} \dot{x} \\ \dot{\beta} \end{pmatrix} = J(q)\dot{q} \in \mathbb{R}^6, \quad \beta \in \Omega \subset \mathbb{R}^3. \quad (5)$$

$\beta \in \Omega \subset \mathbb{R}^3$  is a minimal representation of EF attitude (i.e. of the rotation group  $\text{SO}(3)$ ). In general  $\beta = f(A)$  for some function  $f(\cdot)$  which is many-to-one or undefined if the domain of  $f(\cdot)$  on  $\text{SO}(3)$  is not properly restricted. That is, because  $\text{SO}(3)$  cannot be covered by a single coordinate chart,  $\beta$  is not valid for all possible EF orientations and there will be singularity of attitude representation unless we restrict EF attitude to the region of  $\text{SO}(3)$  for which  $\beta$  is valid [25] [41][42]. This restriction then forces  $\beta$  to be defined in the image of admissible attitudes, namely in some  $\Omega \subset \mathbb{R}^3$ . (It may be true, however, that  $\Omega = \mathbb{R}^3$  as in the case of Euler-Rodriguez parameters where singularity of attitude representation corresponds to  $\|\beta\| = \pi$  [42]). Typical  $\beta$ 's are roll-pitch-yaw angles, axis/angle variables, Euler angles, Euler parameters, and Euler-Rodriguez parameters [25], [41]-[43]. The kinematical relationship between  $\beta$  and  $\omega$  is given by

$$\dot{\beta} = \Pi(\beta)\omega \quad (6)$$

where  $\Pi \in \mathbb{R}^{3 \times 3}$  will lose, rank, i.e. become singular, precisely when  $\beta$  becomes a singular representation of EF attitude. Note from (3)-(6) that

$$J = \begin{pmatrix} I & 0 \\ 0 & \Pi \end{pmatrix} J_0.$$

Generally, the standard Jacobian matrix  $J_0$  will become singular only at a manipulator kinematic singularity, in which case  $J$  will also be singular. Furthermore,  $J$  will be singular when  $\beta = \beta(q)$  gives a singularity of EF attitude representation. This compounds the trajectory planning problem for EF motions, since now we must plan trajectories which avoid manipulator kinematic singularities and also ensure that  $\beta(q) \in \Omega$ .

Henceforth the system (1), (2) or (1), (3) will be said to be exactly externally linearized and decoupled if

$$\dot{V} = u \in \mathbb{R}^6. \quad (7)$$

This is somewhat of an abuse of notation as a consideration of the system (1), (4) shows. For  $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ ,  $\dot{V} = u$  yields

$$\ddot{x} = u_1 \in \mathbb{R}^3, \quad \dot{\omega} = u_2 \in \mathbb{R}^3, \quad \dot{A} = \tilde{\omega} A. \quad (8)$$

Although EF positional dynamics are decoupled and linearized to  $\ddot{x} = u$ , attitude dynamics are nonlinear and given by  $\dot{\omega} = u_2$ ,  $\dot{A} = \tilde{\omega} A$ . Eq. (8) is precisely the sense in which RAC can be said to almost "exactly externally linearize and decouple" attitude error dynamics as was discussed in the introduction. In the case of the system (1), (5),  $\dot{V} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  gives

$$\ddot{x} = u_1 \in \mathbb{R}^3, \quad \ddot{\beta} = u_2 \in \mathbb{R}^3, \quad \beta \in \Omega \subset \mathbb{R}^3, \quad (9)$$

which can indeed be said to be exactly externally linearized and decoupled. Drawbacks to using (9) are that  $\dot{B}$  must always be controlled to remain in  $\Omega$ , trajectories involving  $\dot{B}$  may be difficult to visualize, and the generalized force,  $u_2$ , which drives  $\dot{B}$  may be nonintuitive. On the other hand, it is obvious how to obtain stable attitude tracking from (9). The advantage to using (8) is that  $\omega$  and  $A$  are easily visualized entities, while  $u_2$  is the ordinary torque that we are all familiar with. Fortunately, despite the nonlinear attitude dynamics, it is possible to use (8) to perform EF attitude tracking with asymptotically vanishing attitude error [7] [8].

Note that once (8) is obtained, it is easy to get (9) by use of the relationship (6). If we have  $\dot{\omega} = u$ ,  $\dot{B} = \Pi(\dot{B})\omega$ , and  $\dot{B} \in \Omega$  so that  $\Pi^{-1}(\dot{B})$  exists, then use of

$$\dot{\omega} = u, \quad u = \Pi^{-1}(\dot{B}) (\ddot{B} - \dot{\Pi}(\dot{B})\omega) \quad (10)$$

gives

$$\ddot{B} = \Pi(\dot{B})\dot{\omega} + \dot{\Pi}(\dot{B})\omega = \ddot{u}. \quad (11)$$

Therefore, having (8), we can perform attitude control directly on  $\dot{\omega} = u_2$ ,  $\dot{A} = \ddot{\omega}A$  or we can transform to  $\ddot{B} = \ddot{u}_2$  and then control.

##### 5. Comparison of GCT, NDT, and OSC.

For brevity, we consider the non-redundant manipulator case, taking  $n = 6$  in (1), and we omit derivations. A more detailed discussion is given in [24].

Note that the system (1), (5) can be written as

$$\frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ -M^{-1}V \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix} \tau, \quad y = h(q) \quad (12)$$

or, taking  $Z = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$ ,

$$\frac{d}{dt} Z = A(Z) + B(Z)\tau, \quad y = H(Z) \quad (13)$$

where the definitions of  $A$ ,  $B$ , and  $H$  are obvious. GCT asks: does there exist (i) a nonlinear feedback  $\tau = Q(Z) + B(Z)u$  and (ii) a nonlinear change of basis  $x = X(Z)$  such that (12) is placed into BCF?:

$$\frac{d}{dt} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u \Leftrightarrow \ddot{y} = u. \quad (14)$$

The constructive sufficient conditions of [19]-[22] can be applied and give the following linearizing and decoupling feedback law:

$$\tau = -MJ^{-1} \partial J \dot{q} + V + MJ^{-1}u$$

where

$$\partial J = \left[ \sum_{k=1}^n \frac{\partial J_{ik}}{\partial q_j} \dot{q}_k \right].$$

Although  $\partial J \neq J$ , it is true that  $\partial J \dot{q} = \dot{J} \dot{q}$  giving

$$\tau = -MJ^{-1} \dot{J} \dot{q} + MJ^{-1}u + V. \quad (15)$$

Note that  $J$  must be nonsingular for (15) to exist. This is consistent with the theory of [19]-[12] which provides sufficient conditions for local linearization. Note also that to implement (15), explicit expressions for  $M$ ,  $J^{-1}$ ,  $J$ , and  $V$  are required.

The NDT approach of [11], constructs the linearizing feedback in the following way. For the system (13) define

$$G(Z) = \frac{\partial}{\partial Z} \left( \left[ \frac{\partial}{\partial Z} C(Z) \right] \cdot A(Z) \right), D^*(Z) = G(Z) \cdot B(Z), C^*(Z) = G(A) \cdot A(Z). \quad (16)$$

The use of

$$\tau = -D^{*-1}(Z) C^*(Z) + D^{*-1}(Z)u \quad (17)$$

will transform (12), (13) to  $\ddot{y} = u$ , i.e. to (14).

It is straightforward to show that, for A, B, and C as in (12) and (13), eq. (17) is precisely eq. (15). Note that in (13) we take  $Z = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$  and not  $Z = (q_1, \dot{q}_1, \dots, q_n, \dot{q}_n)^T$ . The latter choice for Z is taken in [11] and serves to obscure the final result - namely that (17) and (15) are equivalent.

Now consider the OSC approach of [12]-[14]. In (1) let  $V = B-G$  where B is the coriolis forces and G the gravity forces. Restrict the domain of the system (1), (5) to ensure that  $h(\cdot)$ , is a bijection (and consequently  $\det J(q) \neq 0$  on this restriction). This restriction means that, as for DGC and NDT, the following gives a local result for external linearization. In [12]-[14], the effective EF dynamics are determined to be

$$\Lambda(y)\ddot{y} + C(y, \dot{y}) = F, \quad \tau = J^T F, \quad \Lambda = J^{-T} M J, \quad C = U - P, \quad (18)$$

$$P = J^{-1} G, \quad U = J^{-T} B - \Lambda \dot{J} \dot{q}, \quad q = h^{-1}(y), \quad \dot{q} = J^{-1} \dot{y}$$

Recall that for the system (1),  $M\ddot{q} + V = \tau$ , the Computed Torque technique is to take  $\tau = Mu + V$ , yielding  $M(\ddot{q}-u) = 0 \Rightarrow \ddot{q} = u$ , since  $M(q) > 0, \forall q$ . Similarly, in (18)  $\Lambda(y) > 0$  for every  $y = h(q)$  where  $q$  is on the restricted domain. Therefore a choice of

$$F = \Lambda(y)u + C(y, \dot{y}), \quad \tau = J^T F \quad (19)$$

in (18) yields  $\Lambda(y)(\ddot{y}-u) = 0 \Rightarrow \ddot{y} = u$ . In this sense the work in [12]-[14] can be viewed as a Generalized Computed Torque technique. From (18) and (19) it is straight forward to determine that  $\tau$  of (19) is exactly  $\tau$  of eq. (15).

#### 6. Derivation of a Feedback Law for Local Exact Decoupled External Linearization and Its Relationship to RAC and GCT.

Recall that the system (1), (7) or (1), (3) is of the form

$$\begin{aligned} M(q)\ddot{q} + V(q, \dot{q}) &= \tau; \quad q \in N^n; \quad \dot{q}, \ddot{q} \in R^n \\ y &= h(q) \in M^m; \quad h(\cdot) \text{ is } C^2; \quad V = J(q)\dot{q} \in R^m; \quad \bar{J}(q) \text{ is } C^1; \\ M(q) &\in R^{n \times n}; \quad M(q) = M(q)^T > 0, \quad q \in N^n \end{aligned} \quad (20)$$

where in general, it may be that  $m \neq n$ ,  $M^m \neq R^m$ ,  $V \neq \dot{y}$ , and  $\bar{J} \neq J = \partial h / \partial q$ .

It is assumed that a necessary and sufficient condition for  $h(q)$  to be onto some neighborhood of  $y = h(q)$  in  $M^m$  is that the mapping  $J(q)$  be onto  $R^m$ , i.e. we assume that  $J(q)$  is onto  $R^m$  if and only if  $J(q) = \partial h(q) / \partial q$  is onto  $T_h(q)M^m = R^m$ . This is a reasonable assumption; for example, when  $M^m = R^m$ ,  $V = \dot{y} = J(q)\dot{q} \in R^m$ , and  $\bar{J} = J = \partial h / \partial q$  this is trivially true. For the case  $y = h(q) = (x, A) \in M^6 = R^3 \times SO(3)$  and  $\bar{J} = J_0$  where  $V = \begin{pmatrix} \dot{x} \\ \omega \end{pmatrix} = J_0(q)\dot{q}$ , the fact that  $\dot{x} \in T_x R^3 = R^3$  and  $A = \tilde{\omega}A \in T_A SO(3)$  means that for  $J_0(q)$  onto, we can fill out a neighborhood of  $(x, A)$  and otherwise we cannot. (A general element of  $T_A SO(3)$  is precisely of the form  $\tilde{\omega}A$ ,  $\tilde{\omega} \in R^{3 \times 3}$  skew-symmetric, so that if  $\omega = \omega(q)$  can be mapped onto  $R^3$ ,  $\tilde{\omega}A$  can be mapped onto  $T_A SO(3)$  [44][47].)

**Definition LEL:** The system (20) can be locally exactly linearized and decoupled (LEL) over an open neighborhood  $B^m(y') \subset M^m$  of  $y' \in h(N^n) \subset M^m$  with the arm in the configuration  $q' \in h^{-1}(y')$  if there is an open neighborhood of  $q'$ ,  $B^n(q') \subset N^n$ , such that  $B^m(y') = h(B^n(q'))$  and if for any  $u \in R^m$  and  $q \in B^n(q')$  there exists

a nonlinear feedback  $\tau = F(q, \dot{q}, u)$  such that  $\dot{V}$ , the velocity associated with  $y = h(q) \in B^m(y')$ , obeys  $\dot{V} = u$ .

Note that for an EF to be LEL at  $y'$  it must be true that  $y'$  be in the range of  $h(\cdot)$ , i.e.  $y'$  must be a physically attainable EF position. Also for a given EF location,  $y' \in h(N^m)$ , a manipulator can physically be in only one of the possible configurations  $h^{-1}(y')$ . Thus we can interpret  $q' \in h^{-1}(y')$  to be the actual physical configuration of a manipulator. If the system (20) is not LEL at  $y'$  in the configuration  $q' \in h^{-1}(y')$  it may be LEL at a different configuration  $q'' \in h^{-1}(y')$ .

**Theorem LEL:** A necessary and sufficient condition for (20) to be LEL at  $y' \in h(N^m)$  in the configuration  $q' \in h^{-1}(y')$  is that  $\bar{J}(q') \in R^{m \times n}$  be onto, which is true iff  $m \leq n$  and  $\text{rank } \bar{J}(q') = m$ . Furthermore, the locally exactly linearizing and decoupling feedback is given by

$$\tau = M(q) \xi + V(q, \dot{q}) \quad (21)$$

where  $\xi$  is any solution to

$$\bar{J}(q) \xi = -\dot{\bar{J}}(q) \dot{q} + u. \quad (22)$$

When  $m = n$  this gives

$$\tau = -M(q) \bar{J}(q)^{-1} \dot{\bar{J}}(q) \dot{q} + M(q) \bar{J}(q)^{-1} u + V(q, \dot{q}). \quad (23)$$

□

**Proof. Necessity:** Suppose that  $\dot{V} = \bar{J}(q') \ddot{q}' + \dot{\bar{J}}(q') \dot{q}' = u$  can be made to hold regardless of the value of  $u \in R^m$ . This means that there must exist  $\ddot{q}' \in R^n$  such that

$$\bar{J}(q') \ddot{q}' = -\dot{\bar{J}}(q') \dot{q}' + u. \quad (24)$$

If  $\bar{J}(q')$  is not onto, then  $\text{Im } \bar{J}(q') \subset R^m$  and  $\text{Im } \bar{J}(q') \neq R^m$ . Let  $u$  be such that  $-\dot{\bar{J}}(q') \dot{q}' + u \notin \text{Im } \bar{J}(q')$ . Then there is no  $\ddot{q}'$  for which (24) holds, yielding a contradiction. **Sufficiency:** By assumption  $\bar{J}(q')$  is full rank and onto  $\Leftrightarrow J(q') = \partial h(q') / \partial q$  is full rank and onto. Since  $\bar{J}$  and  $J$  are  $C^1$ , there exists neighborhoods  $B^m(y')$  and  $B^n(q')$ ,  $y' = h(q')$ , such that  $B^m(y') = h(B^n(q'))$  and such that  $\bar{J}$  is full rank and onto when restricted to  $B^n(q')$ . Now consider any  $q \in B^n(q')$  and its associated  $y = h(q) \in B^m(y')$ . Then,  $\dot{V} = \bar{J}(q) \dot{q} \Rightarrow$

$$\dot{V} = \bar{J}(q) \ddot{q} + \dot{\bar{J}}(q) \dot{q}. \quad (25)$$

Let  $\xi$  be any solution to (22).  $\xi$  is guaranteed to exist since  $\text{Im } \bar{J}(q) = R^m$ . Take  $\tau$  to be (21), then

$$M \ddot{q} + V = \tau = M \xi + V \Rightarrow M(\ddot{q} - \xi) = 0 \Rightarrow \ddot{q} = \xi,$$

which with (22) and (25) gives  $\dot{V} = u$ . □

**Comments:**

1) Note that this result applies to all systems of the form (20), of which rigid link manipulators are a special case.

2) Note that with  $y \in M^m$  and  $\tau \in R^n$ , the fact that we need  $m \leq n$  can be interpreted to mean that there must be least as many inputs as outputs.

3) When  $\bar{J} = J = \partial h / \partial q$ ,  $\dot{V} = \dot{y}$ , and  $m = n$  we have that  $\tau = -MJ^{-1} \dot{J} \dot{q} + MJ^{-1} u + V \Rightarrow \ddot{y} = u$  when  $\det J \neq 0$ . This is the same result provided by GCT, NDT, and OCS as seen in the last section.

4) Note that in the proof we force  $\ddot{q} = \xi$  precisely like  $\ddot{q} = u$  is forced to happen in the Computed Torque method. In fact, for  $y = q$  we have  $J = I$  and  $\dot{J} = 0$  giving  $\xi = u$ . Thus the exact linearizing control of (21), (22) is seen to be a generalization of the Computed Torque method in a somewhat different, and perhaps more illuminating, way than OCS.

Let us consider the case of EF control given by the system (1), (4). Here  $J = J_0$  where  $\dot{V} = \begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} = J_0 \dot{q}$ . In this case, when  $m = n$ , (23) is

$$\tau = -MJ_0^{-1} \ddot{q} + MJ_0^{-1} \dot{u} + V. \quad (26)$$

When  $\det J_0 \neq 0$ , use of  $\tau$  yields  $\begin{pmatrix} \ddot{x} \\ \ddot{\omega} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ . This is precisely RAC [7], [8]. Theorem LEL can be interpreted as an extension of RAC to the redundant arm case which allows for the use of a minimal representation of EF attitude [24]. The more general case  $m \leq n$  is given by

$$\tau = Mu + V, \quad J_0 \xi = -\dot{J}_0 \dot{q} + u. \quad (27)$$

By using the indirect form (27),  $\tau$  can be obtained, after  $\xi$  has been found, by use of the Newton-Euler recursion [45]. Furthermore  $\xi$  can be obtained recursively - either directly [46], or by first recursively obtaining  $J_0$  and  $\dot{J}_0$  and then solving for  $\xi$  by Gaussian Elimination. The major point to be drawn here, is that (27) shows us how to perform exact external linearization without the need for an explicit manipulator model. After exactly linearizing to  $\begin{pmatrix} \ddot{x} \\ \ddot{\omega} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  one can perform EF tracking at this stage [7][8], or one can continue to the form (11) by the use of (10).

When using (26) or (27), the only way that  $\text{rank } J_0 < m$  can occur for  $m \leq n$  is when the manipulator is at a mechanically singular configuration. Recall (section 4) that in the case when a minimal representation of EF attitude is used, the resulting Jacobian matrix  $J$  will be rank deficient not just for a manipulator singularity, but at a configuration which leads to a singularity of attitude presentation. Thus rank deficiency of  $J_0$  is kinematically cleaner to understand. The necessity that  $\text{rank } J_0 = m$  in order to use (26) or (27) allows two obvious, but important statements to be made: i) For a manipulator with a workspace boundary (ignoring joint stops), as in the case of a PUMA-type manipulator, exact linearization at the boundary is impossible; ii) For a nonredundant (6 dof) manipulator with workspace interior singularities, there cannot be exact linearization throughout the workspace interior. For a redundant manipulator with workspace interior singularities, it may be possible to avoid workspace interior configurations which cannot be exactly linearized by the use of self motions as described in [48][49]. This is related to the multiplicity of solutions available for  $\xi$  in (27).

It is interesting to ask just how the control (23) fulfills the aim of GCT as stated in (12)-(14). We have the nonlinear feedback (taking  $\dot{V} = \dot{y}$  and  $J = \bar{J}$ )  $\tau = Q(Z) + B(Z)u = (V - MJ^{-1} \dot{J} \dot{q}) + (MJ^{-1})u$  which when applied to (12), (13) gives

$$\frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & -J^{-1} \dot{J} \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0 \\ J^{-1} \end{pmatrix} u. \quad (28)$$

Consider the local nonlinear change of basis given by

$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} h(q) \\ J \dot{q} \end{pmatrix}; \quad \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} h^{-1}(y) \\ J^{-1} \dot{y} \end{pmatrix}.$$

The fact that  $\dot{y} = J \dot{q}$  and  $\ddot{y} = J \ddot{q} + \dot{J} \dot{q}$  gives

$$\frac{d}{dt} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix} \frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}.$$

Writing (28) as

$$\begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix} \frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & -J^{-1} \dot{J} \end{pmatrix} \begin{pmatrix} h^{-1}(y) \\ J^{-1} \dot{y} \end{pmatrix} + \begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix} \begin{pmatrix} 0 \\ J^{-1} \end{pmatrix} u$$

we obtain the BCF

$$\frac{d}{dt} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u, \quad \Leftrightarrow \ddot{y} = u.$$

Of course we are benefiting from the hindsight provided us by GCT [15]-[19].

## 7. Concluding Remarks

Recognizing the fundamental unity of RAC, GCT, OSC, and NDT [7]-[22] for exact linearization of manipulators, we can focus on their true differences - namely differences in implementation detail and design philosophy. With the awareness that they all produce essentially the same linearizing feedback, we can ask why this particular feedback form is appropriate for manipulator-like systems.

OCS and RAC exploit the specific structure of such systems. Not surprisingly, the solutions arrived at, reflecting the philosophies and implementation perspectives of the researchers involved, are quite distinct in their flavor and presentation. Yet, since the properties specific to manipulator dynamics ultimately forced the solution, they are fundamentally the same. (Actually, apparently only OCS worked with a perspective directed specifically towards decoupled EF motions, RAC is content to stop at a point just shy of the goal. It is also interesting that [12] apparently shows an awareness of the relationship between OCS and RAC, and the degree to which RAC can be said to decouple and linearize EF motions). The important point here is that researchers consciously exploited the specific properties of a system of interest, but without pin-pointing precisely what these properties were which made the system amenable to linearizing control.

GCT and NDT provide techniques for exactly linearizing general smooth affine-in-the-input dynamical systems. These techniques ignore any specific nongeneric structural properties that a system might have and as a consequence the solutions obtained are much less transparent than those of OCS or RAC. The strength of these approaches, particularly GCT, is that they can provide necessary and sufficient conditions for a system to be exactly linearizable and constructive sufficient conditions which produce the linearizing feedback when satisfied. These techniques can be applied to systems which defy our abilities to intuit or comprehend - such as manipulators coupled to complex electromechanical actuation devices. Interestingly, when applied to the problem of manipulator exact linearization the solutions obtained can be shown to be equivalent to those of RAC and OCS. Again the structural properties of the system forced the solution. Once a solution is known to exist, it is reasonable to attempt to produce it from more physical arguments knowing now that the search is not fruitless. This leads to a reexamination of OCS and RAC.

The work of [17]-[22] stresses a perspective which serves to enable a clearer comparison between competing techniques for external linearization: Place the system in a standard linear canonical form before additional control efforts are made - this ensures that the process of linearizing the system is not mixed up with, and confused with, the process of stabilizing and controlling it. This perspective greatly aided the comparison of GCT, OCS, RAC, and NDT which resulted in [24]. In turn, this comparison focuses attention on the structural properties of manipulators.

Much current research makes it apparent that systems dynamically similar to rigid link manipulators have important structural properties which can be exploited to achieve results which are quite strong when compared to those available for general smooth affine-in-the-inputs nonlinear systems [25]-[40]. Here we have seen that exploiting the nongeneric second order form of system (1) with an everywhere positive definite mass matrix and a  $\dot{C}^2$  locally onto readout map enables a simple form for the linearizing feedback.

## 8. Acknowledgments

This work was done at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

## 9. References

- [1] Paul, R. C., "Modeling, Trajectory Calculation and Servoing of a Computer Controlled arm.", Stanford A. I. Lab., Stanford U., A. I. Memo 177, Nov. 1972.
- [2] Markiewicz, B. R., "Analysis of the Computed Torque Drive Method and Comparison with Conventional Position Servo for a Computer-Controlled Manipulator." JPL TM 33-601, March 1973.
- [3] Bejczy, A. K., "Robot Arm Dynamics and Control," JPL TM 33-669, Feb. 1974.
- [4] Hemami, H. and Camana, P. C., "Nonlinear Feedback in Simple Locomotion Systems", IEEE Trans. Automatic Control, 1976, pp. 855-860.
- [5] Raibert, M. H. and Horn, B. K., "Manipulator Control Using the Configuration Space Method", Industrial Robot, 5, pp. 69-73, 1978.
- [6] Hewit, J. R. and Padovan, J., "Decoupled Feedback Control of Robot and Manipulator arms", Proc. 3rd CISM-IFTOMM Symp. on Theory and Practice of Robot Manipulators, Udine, Italy, pp. 251-266, Sept. 1976.
- [7] Paul, R., Luh, J. et al., "Advanced Industrial Robot Control Systems", Purdue U., TR-EE 78-25, May 1978.
- [8] Luh, J., Walker, M. and Paul, R., "Resolved - Acceleration Control of Mechanical Manipulators", IEEE Trans. Aut. Control, AC-25, pp. 468-474, 1980.
- [9] Freund, E., "A Nonlinear Control Concept for Computer Controlled Manipulators", Proc IFAC Symp. Multivariable Technological Systems, pp. 395-403, 1977.
- [10] Freund, E. and Syrbe, M., "Control of Industrial Robots by Means of Microprocessors", IRIA Conf. Lecture Notes on Information Sciences, Amsterdam, Springer-Verlag, pp. 167-185, 1976.



- [11] Freund, E., "Fast Nonlinear Control with Arbitrary Pole-Placement for Industrial Robots and Manipulators", International J. Robotics Research, 3, pp. 76-86, 1982.
- [12] Khatib, O., Commande dynamique dans l'espace operational des robots manipulateurs en presence d'obstacles, Engineering Doctoral Thesis, No. 37, ENSAE, Toulouse, 1980.
- [13] Khatib, O., "Dynamic Control of Manipulators in Operational Space", 6th CISM-IFTOMM, 1983.
- [14] Khatib, O., "The Operational Space Formulation in the Analysis, Design, and Control of Robot Manipulators", 3rd Int'l Symp. Robotics Research, pp. 103-110, 1985.
- [15] Isidori, A. and Ruberti, A., "On the Synthesis of Linear Input-output Responses for Nonlinear Systems", Systems and Control letters, 4, pp 17-22, 1984.
- [16] Isidori, A., "The Matching of a Prescribed Linear Input-Output Behavior in a Nonlinear System", IEEE Trans. Aut. Control, Vol. AC-30, pp. 258-265, 1985.
- [17] Cheng, D., Tarn, T. J., and Isidori, A., "Global External Linearization of Nonlinear Systems Via Feedback", IEEE Trans. Automatic Control.
- [18] Isidori, A., Nonlinear Control Systems: An Introduction, Springer-Verlag, 1985.
- [19] Chen, Y., Nonlinear Feedback and Computer Control of Robot Arms, D. Sc. Dissertation, Dept. Systems Science and Math, Washington U., St. Louis MO, 1984.
- [20] Tarn, T. J., Bejczy, A. K. et al., "Nonlinear Feedback in Robot Arm Control", Proc. 23rd CDC, pp. 736-751, 1984.
- [21] Bejczy, A. K., Tarn, T. J., and Chen, Y. L., "Robot Arm Dynamic Control by Computer, 1985 IEEE ICRA", pp. 960-970.
- [22] Tarn, T. J., Bejczy, A. K., and Yun, X., "Coordinated Control of Two Robot Arms", 1986 IEEE ICRA, pp. 1193-1202.
- [23] Brady, M. et al. Robot Motion Planning and Control, MIT press, 1984.
- [24] Kreutz, K., "On Nonlinear Control for Decoupled Exact External Linearization of Robot Manipulators", in Recent Trends in Robotics: Modeling, Control, and Education, Ed. by M. Jamshidi et al. pp. 199-212, North-Holland, 1986.
- [25] Meirovitch, L., Methods of Analytical Dynamics, McGraw-Hill, New York 1970.
- [26] Gantmacher, F., Lectures in Analytical Mechanics, MIR publishers, Moscow, 1975.
- [27] Koditschek, D., "Natural Control of Robot Arms", Center for Systems Science Report, Dept. of EE, Yale U., 1985.
- [28] Koditschek, D., "High Gain Feedback and Telerobotic Tracking", This Workshop.
- [29] Asada, H. and Slotine, J., Robot Analysis and Control, Wiley, New York, 1986.
- [30] Pringle, R. Jr., "On the Stability of a Body with Connected Moving Parts", AIAA Journal, 4, pp. 1395-1404, 1966.
- [31] Takegaki, M. and Arimoto, S., "A New Feedback Method for Dynamic Control of Manipulators", J. of Dyn. Sys. Meas. Control, 102, pp. 119-125, 1981.
- [32] Arimoto, S. and Miyazaki, F., "Stability and Robotness of PID Feedback Control for Robot Manipulators of Sensory Capacity", First Int'l. Symp. Robotics Research, pp. 783-799, 1983.
- [33] Arimoto, S. and Miyazaki, F., "Stability and Robustness of PD Feedback Control With Gravity Compensation for Robot Manipulators", Robotics: Theory and Practice, DSC-Vol. 3, pp. 67-72, 1986.
- [34] Wen, J. T. and Bayard, D.S., "Simple Robust Control Laws for Robotic Manipulators - Part I: Nonadaptive Case", This Workshop.
- [35] Bayard, D. S. and Wen, J. T., "Simple Robust Control Laws for Robotic Manipulators - Part II: Adaptive Case", This Workshop.
- [36] Slotine, J. and Li, W., "On the Adaptive Control of Robot Manipulators", Robotics: Theory and Practice, DSC-Vol. 3, pp. 51-56, 1986.
- [37] Paden, B., and Slotine, D., "PD + Robot Controllers: Tracking and Adoptive Control", 1987 IEEE Int'l Conf. Robotics and Automation Raleigh, 1987 (to be presented).
- [38] Sontag, E. and Sussman, H., "Time-Optimal Control of Manipulators," IEEE 1986 Int'l. Conf. Robotics and Automation, San Francisco, 1986, pp. 1692-1697.

- [39] Sontag, E. and Sussman, H., "Remarks on the Time-Optimal Control of Two-link Manipulators", Proc. 24th IEEE CDC, pp. 1643-1652, 1985.
- [40] Wen, J., "On Minimum Time Control for Robotic Manipulators", in Recent Trends in Robotics: Modeling, Control, and Education, Ed. by M. Jamahidi et al., pp. 283-292, North-Holland, 1986.
- [41] Stuelpnagel, J., "On the Parameterization of the Three Dimensional Rotation Group", SIAM Review, 6, 422-430, 1964.
- [42] Hughes, P.C., Spacecraft Attitude Dynamics, Wiley, New York, 1986.
- [43] Craig, J., Introduction to Robotics, Addison-Wesley, 1986.
- [44] Boothby, W., An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd Edition, Academic Press, Orlando, 1986.
- [45] Luh, J. Y. S., Walker, M. W., and Paul R. P., "On-line Computational Scheme for Mechanical Manipulators", J. Dynamical Systems, Measurement, and Control, 102, 1980, pp. 69-76.
- [46] Hollerbach, J. M. and Sahar, G., "Wrist-Partitioned Inverse Kinematic Accelerations and Manipulator Dynamics", Int'l. J. Robotics Research, 2, pp. 61-76, 1983.
- [47] Von Westenholz, C., Differential Forms in Mathematical Physics, North-Holland, Amsterdam, 1986.
- [48] Hollerbach, J., "Optimum Kinematic Design for a Seven Degree of Freedom Manipulator", 2nd Int'l Symp. Robotics Research, 1984.
- [49] Baillieul, J. et al., "Programming and Control of Kinematically Redundant Manipulators", PROC. 23rd CDC, pp. 768-774, 1986.